

CLARKSON-ERDÖS-SCHWARTZ THEOREM ON A SECTOR

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ABSTRACT. We prove a Clarkson-Erdős-Schwartz type theorem for the case of a closed sector in the plane. Concretely, we get some sufficient conditions for the incompleteness and minimality of a Müntz system $E(\Lambda) = \{z^{\lambda_n} : n = 0, 1, \dots\}$ in the space H_α , where $H_\alpha = A(I_\alpha)$, $I_\alpha = \{z \in \mathbb{C} : |\arg(z)| \leq \alpha \text{ and } |z| \leq 1\}$ and $A(K) = C(K) \cap H(\text{Int}[K])$ denotes the space of continuous functions on the compact set K which are analytic in the interior of K . Furthermore, we prove that, if $\text{span}[E(\Lambda)]$ is not dense in H_α then all functions $f \in \overline{\text{span}}[E(\Lambda)]$ can be analytically extended to the interior of the sector I_π .

1. INTRODUCTION

Suppose $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ is a sequence of positive real numbers arranged for convenience in non-decreasing order:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \text{ and } \delta(\Lambda) = \inf\{\lambda_{n+1} - \lambda_n : n \geq 0\}.$$

$E(\Lambda) = \{z^{\lambda_n} = \exp\{\lambda_n \log z\} : n = 0, 1, 2, \dots\}$ is called a Müntz system, $\text{span}[E(\Lambda)]$ denotes the linear span of the Müntz system. The elements of the set $\text{span}[E(\Lambda)]$ are called the Müntz polynomial or the Λ -polynomials [4]. Let K be a compact set in the plane, and let $C(K)$ be the space of all continuous functions on K , equipped with the uniform norm. The famous Müntz theorem ([4] and [18]) states that $\text{span}[E(\Lambda)]$ is dense in $C([0, 1])$ if and only if $\sum_{n=1}^{\infty} 1/\lambda_n$ diverges. Moreover, If the set $\text{span}[E(\Lambda)]$ is not a dense subspace of $C[0, 1]$, it is natural to ask for a characterization of the elements of its topological closure. This problem was solved by Clarkson and Erdős [5] for the case of integer exponents Λ , they proved that if $\sum_{n=1}^{\infty} 1/\lambda_n$ converges, the elements in the closure of $\text{span}[E(\Lambda)]$ can be extended analytically throughout to the unit disc with a series expansion of the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^{\lambda_k}, \quad 0 \leq x < 1. \quad (1)$$

This same question was also tackled by L.Schwartz [16] for certain strictly increasing sequences of exponents (he assumed $\delta(\Lambda) > 0$) and by Borwein [4] and Érdelyi [6]. Nowadays these results are referenced under the join name of Clarkson-Erdős-Schwartz Theorem [1]. On the other hand, if K is a compact in the plane whose complement is connected, Mergelyan's theorem [15] claims that the space of complex polynomials is a dense set of $C(K)$. It is a nontrivial problem to establish a

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Clarkson-Erdős-Schwartz Theorem on a closed sector in the plane. The aim of this paper answer to this problem. First we introduce some notations and definitions. Let B be a Banach space. If $E = \{e_k : k = 1, 2, \dots\} \subset B$, let $\mathbf{span}[E]$ denote the subspace of B , consisting of all finite linear combinations of E and let $\overline{\mathbf{span}}[E]$ be the closure of $\mathbf{span}[E]$ in B . The set E is said to be *incomplete* in B [17] if $\overline{\mathbf{span}}[E]$ does not coincide with the whole B . The set E is called to be *minimal* in B [17] if no element of E belongs to the closure of the vector subspace generated by the other elements of E , i.e., for all $e \in E$, $e \notin \overline{\mathbf{span}}[E - \{e\}]$. The minimality of the E is equivalent to the existence of $\{f_n : n = 1, 2, \dots\}$ conjugate functionals in the dual Banach space B^* of B . By $\{f_n\}$ been conjugate with respect to $\{e_n\}$ we means that $f_n(e_m) = \delta_{nm}$ for all n, m , where δ_{nm} is well know Kronecker's symbol. $\{f_n : n = 1, 2, \dots\}$ is also called a biorthogonal system of E . It follows that if E is minimal, each $x \in \overline{\mathbf{span}}E$ has a unique formal E -expansion $\sum x_n e_n$ [17], where $x_n = f_n(x)$.

In this paper, we particularize B to be the Banach space H_α consisting of all functions $f(z)$ which are continuous on the closed sector $I_\alpha = \{z = re^{i\theta} : 0 \leq r \leq 1, |\theta| \leq \alpha\}$ ($0 \leq \alpha < \pi$), analytic in $\mathbf{int}[I_\alpha]$. The norm of f is given by

$$\|f\| = \max\{|f(z)| : z \in I_\alpha\}.$$

If the Banach space H_α is replaced by the Fréchet space F_α , which consists of all functions analytic in the sector $\mathbf{int}[I_\alpha]$, under the compact topology (uniform convergence on each compact subset of $\mathbf{int}[I_\alpha]$), Khabibullin [8], Rubel [14] and Malliavin [13] have proved that $\mathbf{span}[E(\Lambda)]$ is not dense in F_α if and only if there are $b \in (0, \frac{\pi}{\alpha})$ and M_b such that

$$\lambda(y) - \lambda(x) \leq b \log y - b \log x + M_b \quad (y > x \geq 1),$$

where the characteristic logarithm $\lambda(t)$ is defined by ([13] and [14])

$$\lambda(t) = \sum_{0 < \lambda_n \leq t} \lambda_n^{-1}. \quad (2)$$

Inspired by the method of Khabibullin [8], Anderson [2], Rubel and Malliavin, we obtain the following result.

Theorem 1. *Let $\alpha \in [0, \pi)$ and $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ be a sequence satisfying $\delta(\Lambda) > 0$ and assume that there exists a decreasing function $\varepsilon(x)$ on $[0, \infty)$ such that $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$ and such that*

$$\lambda(y) - \lambda(x) \leq \frac{\alpha}{\pi} \log y - \frac{\alpha}{\pi} \log x + \varepsilon(x), \quad (y > x \geq 1) \quad (3)$$

Then $E(\Lambda)$ is minimal and $\overline{\mathbf{span}}[E(\Lambda)]$ does not contain the function z^λ for $\lambda \notin \Lambda$, $\operatorname{Re} \lambda > 0$, and each function f in $\overline{\mathbf{span}}(E(\Lambda))$ can be extended analytically throughout the region $\mathbf{int}(I_\pi)$ with a series expansion of the form (1).

Remark 1 If $\lambda(t)$ is bounded on $[1, \infty)$, then the function $\varepsilon(x) = \lambda(\infty) - \lambda(x)$ is decreasing, $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$ and satisfies (3) with $\alpha = 0$. So we have the following Corollary which can be found in [4] (p.178).

Corollary 1. *Let $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ be a sequence satisfying $\delta(\Lambda) > 0$. If $\lambda(t)$ is bounded on $[1, \infty)$, then $E(\Lambda) \subset H_0$ is minimal and each function f in $\overline{\mathbf{span}}[E(\Lambda)]$ can be extended analytically throughout the region $\mathbf{int}[I_\pi]$ with a series expansion of the form (1). If $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ is a sequence of distinct*

positive integers and $\lambda(t)$ is bounded on $[1, \infty)$, then each function f in $\overline{\text{span}}[E(\Lambda)]$ can be extended analytically throughout the open unit disk. Therefore, Theorem 1 is a generalization of the Clarkson-Erdős-Schwartz Theorem to a sector.

2. LEMMAS AND PROOFS

In order to prove our main result, we need the following technical lemmas. The following Lemma 1 can be seen from [3] and [12].

Lemma 1 (Fuchs' Lemma). *If Λ is a sequence of positive numbers satisfying $\delta(\Lambda) > 0$, then the function*

$$G(z) = \prod_{n=1}^{\infty} \left(\frac{\lambda_n - z}{\lambda_n + z} \right) \exp \left(\frac{2z}{\lambda_n} \right) \quad (4)$$

is a meromorphic function and satisfies the following inequalities:

$$|G(z)| \leq \exp\{x\lambda(|z|) + Ax\}, \quad z \in \mathbb{C}_+, x \geq 0, \quad (5)$$

$$|G(z)| \geq \exp\{x\lambda(|z|) - Ax\}, \quad z \in C(\Lambda, \delta_0), \quad (6)$$

where $4\delta_0 = \delta(\Lambda)$ and

$$C(\Lambda, \delta_0) = \{z \in \mathbb{C}_+ : |z - \lambda_n| \geq \delta_0, n = 1, 2, \dots\}. \quad (7)$$

Lemma 2. *Let $\varepsilon(x)$ be a positive decreasing function on $[0, \infty)$, $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ and $\Lambda' = \{\lambda'_n : n = 1, 2, \dots\}$ sequences of positive numbers satisfying $\delta(\Lambda) > 0$ and $\delta(\Lambda') > 0$, respectively. If*

$$\lambda(y) - \lambda(x) \leq \lambda'(y) - \lambda'(x) + \varepsilon(x), \quad y > x \geq 0, \quad (8)$$

then there exist a constant A_1 and a subsequence $\Lambda^ = \{\lambda_n^* : n = 1, 2, \dots\}$ of the sequence $\Lambda' = \{\lambda'_n : n = 1, 2, \dots\}$ such that*

$$|\lambda(x) + \lambda^*(x) - \lambda'(x) - A_1| \leq \varepsilon(x) + x^{-1}, \quad x > 0. \quad (9)$$

Proof of Lemma 2. Similar to the proof in [13, p.181-182] and [14, p.148-149], we define

$$\varphi(x) = \inf\{\lambda'(s) - \lambda(s) : s \geq x\}.$$

It follows from (8) that $\varphi(x) \geq \lambda'(x) - \lambda(x) - \varepsilon(x)$.

Now $\varphi(x)$ is constant except for possible jumps at the jumps of $\lambda'(x)$. Let a be a point of discontinuity of φ . Then, the left limit of φ at a is $\varphi(a-0) = \lambda'(a-) - \lambda(a-0)$ and the right limit of φ at a is $\varphi(a+0) = \varphi(a) \leq \lambda'(a) - \lambda(a)$. We denote by $\Delta\varphi(a) = \varphi(a+0) - \varphi(a-0)$, the jump of φ at a . Then

$$\Delta\varphi(a) \leq \Delta\lambda'(a) - \Delta\lambda(a) \leq \Delta\lambda'(a). \quad (10)$$

Therefore, there exists a sequence $\Lambda^* = \{\lambda_n^* : n = 1, 2, \dots\}$ of positive numbers whose counting function ([13] and [14]) is $\Lambda^*(t) = [\Phi(t)]$, where $[x]$ denotes the integral part of x ,

$$\Phi(t) = \int_0^t s \, d\varphi(s) \quad \text{and} \quad \Lambda^*(t) = \sum_{\lambda_n^* \leq t} 1.$$

The characteristic logarithm $\lambda^*(t)$ of the sequence Λ^* is constant except possibly the jumps of $\varphi(t)$, and we have $\Delta\lambda^*(a) < a^{-1} + \Delta\varphi(a)$. Using (10), we get $\Delta\lambda^*(a) <$

$a^{-1} + \Delta\lambda'(a)$. Furthermore, $a\Delta\lambda^*(a)$ and $a\Delta\lambda'(a)$ must be integers, so $\Delta\lambda^*(a) \leq \Delta\lambda'(a)$ and this means that Λ^* is a subsequence of Λ' . Now,

$$\varphi(x) - \varphi(0) = \int_0^x s^{-1} d\Phi(s) \quad \text{and} \quad \lambda^*(x) = \int_0^x s^{-1} d[\Phi(s)].$$

An integration by parts shows that

$$\lambda^*(x) - \varphi(x) = A_1 + \varepsilon_2(x),$$

where

$$\varepsilon_2(x) = \int_x^\infty (\Phi(s) - [\Phi(s)]) \frac{ds}{s^2} - x^{-1}(\Phi(x) - [\Phi(x)])$$

and

$$A_1 = \int_0^\infty (\Phi(s) - [\Phi(s)]) \frac{ds}{s^2} - \varphi(0).$$

We now define

$$\varepsilon_1(x) = \lambda(x) + \lambda^*(x) - \lambda'(x) - A_1,$$

It is clear that $|\varepsilon_2(x)| \leq x^{-1}$, so by the definition of $\varphi(x)$, $\varepsilon_1(x) \leq \varepsilon_2(x) \leq x^{-1}$. (8) is simply another way of saying that

$$\varepsilon_1(x) \geq \varepsilon_2(x) - \varepsilon(x) \geq -x^{-1} - \varepsilon(x).$$

This proves (9).

Lemma 3. *Let $b \geq 0$ and let $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ be a sequences of positive real numbers satisfying $\delta(\Lambda) > 0$. If there exists a constant A_2 such that*

$$\lim_{x \rightarrow \infty} |\lambda(x) - b \log^+ x - A_2| = 0, \quad (11)$$

then the function

$$g_0(z) = \frac{G(z)e^{-a_0 z}}{\Gamma(\frac{1}{2} + 2bz)}, \quad (12)$$

is meromorphic and satisfies

$$\limsup_{x \rightarrow \infty} x^{-1} \log |g_0(x)| = 0, \quad (13)$$

$$\lim_{x \in C(\Lambda, \delta_0), x \rightarrow \infty} x^{-1} \log |g_0(x)| = 0, \quad (14)$$

and

$$\lim_{k \rightarrow \infty} \lambda_k^{-1} \log |g'_0(\lambda_k)| = 0, \quad (15)$$

where $\Gamma(z)$ is the Euler Gamma function, $G(z)$ is defined by (4), $a_0 = 2A_2 - 2b \log(2b)$ ($a_0 = 2A_2$, if $b = 0$) and $C(\Lambda, \delta_0)$ is defined by (7).

Proof of Lemma 3. The main method of the proof is based on the use of the following function used by Malliavin [11]

$$\psi(s) = 2 + s \log \left| \frac{s-1}{s+1} \right|.$$

The function $\psi(s)$ is decreasing on $[0, 1)$, increasing on $(1, \infty)$ and there exists $s_0 \in (\frac{5}{6}, \frac{6}{7})$ such that $\psi(s_0) = 0$. Thus $\psi(s)$ is negative on $(s_0, 1) \cup (1, \infty)$. Since $\delta(\Lambda) > 0$, $\sum_{n=1}^\infty |\lambda_n|^{-2}$ converges. Thus $G(z)$ defined by (4) is the quotient of convergent canonical products. As a result, the product (4) defines a meromorphic function

in the complex plane \mathbb{C} , which has zeros at each point λ_n . Writing $\log |G(x)|$ as a sum of logarithms, and that sum as a Stieljes integral, we get

$$\log |G(x)| = x \int_0^\infty \psi\left(\frac{t}{x}\right) d\lambda(t).$$

Let

$$k(x) = \lambda(x) - b \log^+ x - A_2, \quad \varepsilon(x) = \sup\{|k(y)| : y \geq x\}$$

and

$$\varepsilon_3(x) = - \int_0^x \log \left| \frac{1-t}{1+t} \right| dt.$$

Then the function $\varepsilon_3(x)$ is continuous on $[0, \infty)$, increasing and positive on $(0, \infty)$, convex on $[0, 1]$ and concave on $[1, \infty)$. Thus $x^{-1}\varepsilon_3(x)$ is increasing on $(0, 1]$ and decreasing on $[1, \infty)$, so $\sup\{x^{-1}\varepsilon_3(x) : x > 0\} = \varepsilon_3(1) = 2 \log 2 < 3$. An easy calculation shows that

$$\int_0^\infty \psi\left(\frac{t}{x}\right) d \log^+ t = \int_{1/x}^{+\infty} \psi(t) \frac{dt}{t} = 2 \log x - 2 + \varepsilon_3(x^{-1}),$$

and the Gamma function $\Gamma(z)$ satisfies

$$\log \left| \Gamma\left(\frac{1}{2} + z\right) \right| = x \log \left| z + \frac{1}{2} \right| - \left| y \arg\left(z + \frac{1}{2}\right) \right| - x + c_1(z), \quad (16)$$

where $c_1(z)$ satisfies $|c_1(z)| \leq 10$ for $x = \operatorname{Re} z \geq 0$. By the choice of a_0 , we obtain

$$x^{-1} \log |g_0(x)| = I_1(x) - 2A_2 + \int_0^\infty \psi\left(\frac{t}{x}\right) dk(t), \quad (17)$$

where function $I_1(x) = b\varepsilon_3(x^{-1}) - x^{-1}c_1(2bx)$ satisfies

$$\lim_{x \rightarrow \infty} |I_1(x)| = 0.$$

Since $k(x)$ has a jump at each point λ_n and $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$, we can assume, without loss of generality, that $1 \geq 6\varepsilon(x) > 0$ for $x \geq 0$ (if not, replaced $\varepsilon(x)$ by $\min\{\frac{1}{6}, \varepsilon(x)\}$). Let $a(x) = 1 + \varepsilon(\frac{6x}{7})$. If we split the range of the integral in (17) into the ranges $(0, \frac{x}{a(x)})$, $[\frac{x}{a(x)}, xa(x)]$ and $[xa(x), \infty)$, integration by parts in $(0, \frac{x}{a(x)})$ and $[xa(x), \infty)$, respectively, then $x^{-1} \log |g_0(x)|$ can be written in the form

$$x^{-1} \log |g_0(x)| = \sum_{j=1}^8 I_j(x),$$

where

$$\begin{aligned} I_2(x) &= k\left(\frac{x}{a(x)}\right) \psi\left(\frac{1}{a(x)}\right) - k(xa(x))\psi(a(x)); \\ I_3(x) &= - \int_0^{\frac{6x}{7}} k(t) \psi'\left(\frac{t}{x}\right) \frac{dt}{x}; \quad I_4(x) = - \int_{\frac{7x}{6}}^\infty k(t) \psi'\left(\frac{t}{x}\right) \frac{dt}{x}; \\ I_5(x) &= - \int_{\frac{6x}{7}}^{\frac{x}{a(x)}} k(t) \psi'\left(\frac{t}{x}\right) \frac{dt}{x}; \quad I_6(x) = - \int_{xa(x)}^{\frac{7x}{6}} k(t) \psi'\left(\frac{t}{x}\right) \frac{dt}{x}; \\ I_7(x) &= -b \int_{\frac{x}{a(x)}}^{xa(x)} \psi\left(\frac{t}{x}\right) d \log^+ t \quad \text{and} \quad I_8(x) = \int_{\frac{x}{a(x)}}^{xa(x)} \psi\left(\frac{t}{x}\right) d\lambda(t). \end{aligned}$$

Next we shall show that

$$\lim_{x \rightarrow \infty} |I_j(x)| = 0, j = 2, 3, \dots, 7. \quad (18)$$

Since $1 < a(x) = 1 + \varepsilon \left(\frac{6x}{7} \right) \leq \frac{7}{6}$,

$$0 \leq -\psi \left(\frac{t}{x} \right) \leq -\log \varepsilon \left(\frac{6x}{7} \right) \quad \left(t \in \left[\frac{6x}{7}, \frac{x}{a(x)} \right] \right)$$

and

$$0 \leq -\psi \left(\frac{t}{x} \right) \leq -2 \log \varepsilon \left(\frac{6x}{7} \right) \quad (t \in [xa(x), \infty)),$$

we see that

$$\begin{aligned} |I_2(x)| &\leq -3\varepsilon \left(\frac{6x}{7} \right) \log \varepsilon \left(\frac{6x}{7} \right); \\ |I_4(x)| &\leq -\varepsilon(x) \int_{\frac{7x}{6}}^{\infty} \psi' \left(\frac{t}{x} \right) \frac{dt}{x} = -\varepsilon(x) \psi \left(\frac{7}{6} \right); \\ |I_5(x)| &\leq -\varepsilon \left(\frac{6x}{7} \right) \int_{\frac{6x}{7}}^{\frac{x}{a(x)}} \psi' \left(\frac{t}{x} \right) \frac{dt}{x} \leq -2\varepsilon \left(\frac{6x}{7} \right) \log \varepsilon \left(\frac{6x}{7} \right); \\ |I_6(x)| &\leq \varepsilon \left(\frac{6x}{7} \right) \int_{xa(x)}^{\frac{6x}{7}} \psi' \left(\frac{t}{x} \right) \frac{dt}{x} \leq -2\varepsilon \left(\frac{6x}{7} \right) \log \varepsilon \left(\frac{6x}{7} \right). \end{aligned}$$

These prove that (18) hold for $j = 2, 4, 5, 6$. Also for $x > 1$,

$$\begin{aligned} |I_3(x)| &\leq -\varepsilon(0) \int_0^{\frac{\sqrt{x}}{2}} \psi' \left(\frac{t}{x} \right) \frac{dt}{x} - \varepsilon \left(\frac{\sqrt{x}}{2} \right) \int_{\frac{\sqrt{x}}{2}}^{\frac{6x}{7}} \psi' \left(\frac{t}{x} \right) \frac{dt}{x} \\ &\leq \frac{\varepsilon(0)}{\sqrt{x}} + 2\varepsilon \left(\frac{\sqrt{x}}{2} \right), \end{aligned}$$

so (18) also holds for $j = 3$. Since $\frac{1}{a(x)} \geq \frac{6}{7} > s_0$, so

$$0 \leq I_7(x) \leq b\varepsilon \left(\frac{6x}{7} \right) \int_{\frac{x}{a(x)}}^{xa(x)} \left(-\frac{t}{x} \log \left| \frac{t}{x} - 1 \right| \right) d \log^+ t$$

and

$$0 \leq -I_8(x) \leq \int_{\frac{x}{a(x)}}^{xa(x)} \left(-\frac{t}{x} \log \left| \frac{t}{x} - 1 \right| \right) d\lambda(t).$$

Therefore

$$0 \leq I_7(x) \leq 2b\varepsilon \left(\frac{6x}{7} \right) \int_0^{\varepsilon \left(\frac{6x}{7} \right)} (-\log s) ds \leq -4b \left(\varepsilon \left(\frac{6x}{7} \right) \right)^2 \log \varepsilon \left(\frac{6x}{7} \right).$$

Hence (18) holds for $j = 7$. Finally,

$$0 \leq -I_8(x) \leq x^{-1} \sum_{\frac{x}{a(x)} < \lambda_n \leq xa(x)} (\log(3x) - \log |\lambda_n - x|).$$

let $\Lambda(t) = \sum_{\lambda_n \leq t} 1$ be the counting function of Λ [3], then for $x \in C(\Lambda, \delta_0)$, we have $|\lambda_n - x| \geq |n - \Lambda(x)|\delta_0$. Let

$$n_1(x) = \max \left\{ \Lambda(xa(x)) - \Lambda(x), \Lambda(x) - \Lambda \left(\frac{x}{a(x)} \right) \right\},$$

then, for $x \in C(\Lambda, \delta_0)$,

$$-I_8(x) \leq \frac{2}{x} \left(n_1(x) \log \left(\frac{3x}{\delta_0} \right) - \log n_1(x)! \right).$$

By $e^n n! \geq n^n (n \geq 1)$,

$$-I_8(x) \leq \frac{2}{x} n_1(x) \left(\log \left(\frac{3ex}{\delta_0} \right) - \log n_1(x) \right).$$

Since the inequalities

$$\Lambda(R) - \Lambda(r) \leq R(\lambda(R) - \lambda(r)) \leq 2R\varepsilon(r) + bR \log \frac{R}{r}$$

hold for $R > r$, we see that

$$n_1(x) \leq 2xa(x)\varepsilon \left(\frac{6x}{7} \right) + bxa(x) \log a(x) \leq 4x(1+b)\varepsilon \left(\frac{6x}{7} \right).$$

The function $t(\log a - \log t)$ is increasing on $(0, ae^{-1})$ ($a > 0$) and there is $x_0 > 1$ such that $9\delta_0(1+b)\varepsilon \left(\frac{6x}{7} \right) \leq 3$, we see that

$$-I_8(x) \leq -18(1+b)\varepsilon \left(\frac{6x}{7} \right) \log \left(\delta_0(1+b)\varepsilon \left(\frac{6x}{7} \right) \right), \quad x \geq x_0.$$

These prove that (13) and (14) hold. Similarly, (15) can also be proved. This completes the proof of Lemma 3.

3. PROOF OF THEOREM

Proof . We can assume that $\alpha > 0$ in the proof of Theorem. It is a consequence of the Hahn-Banach theorem [15] that $\overline{\text{span}}[E(\Lambda)] \neq H_\alpha$ if and only if there exists a bounded linear functional T on H_α with $\|T\| = 1$ which vanishes on all of $E(\Lambda)$. Since every bounded linear functional on H_α is given by integration with respect a complex Borel measure on Λ_α . So we shall construct a bounded linear functional T on H_α such that

$$T(\zeta^z) = g(z) = \frac{z^2 G(z) e^{-Az}}{\Gamma\left(\frac{1}{2} + \frac{2}{\pi}\alpha z\right)(1+z)^4},$$

where $\Gamma(z)$ is the Euler Gamma function, $G(z)$ is defined by (4) and A is a sufficient large positive constant. The function $g(z)$ is analytic in the right half plane \mathbb{C}_+ . Moreover, since $G(z)G(-z) \equiv 1$ and $\Gamma(z)\Gamma(1-z)\sin(\pi z) \equiv \pi$, it follows from Lemma 3, (16) and Cauchy's formula for $g'(z)$ and $g''(z)$ that

$$|g(z)| + |g'(z)| + |g''(z)| \leq \frac{Ae^{\alpha|y|}}{1+|z|^2} \quad (x \geq 0) \quad (19)$$

holds for a sufficient large positive constant A . Fix z so that $x > 0, y > 0$, and consider the Cauchy's formula for $g(z)e^{\alpha zi}$, where the path of integration consists of the quadrant circle with center at 0, radius $R > 1 + |z|$ from R to iR , followed by the interval from iR to 0 and by the interval from 0 to R . The integral over the quadrant circle tends to 0 as $R \rightarrow \infty$, so we are left with

$$g(z)e^{i\alpha z} = \frac{1}{2\pi i} \int_0^{+\infty} \frac{g(t)e^{i\alpha t}}{t-z} dt - \frac{i}{2\pi i} \int_0^{+\infty} \frac{g(it)e^{i\alpha it}}{it-z} dt \quad (20)$$

and similarly, fix z so that $x > 0, y > 0$, and consider the Cauchy formula for $g(z)e^{-i\alpha z}$, where the path of integration consists of the lower quadrant circle with

center at 0, radius $R > 1 + |z|$ from $-iR$ to R , followed by the interval from R to 0 and by the interval from 0 to $-iR$. The integral over the lower quadrant circle tends to 0 as $R \rightarrow \infty$, so we are left with

$$0 = \frac{-1}{2\pi i} \int_0^{+\infty} \frac{g(t)e^{-iat}}{t-z} dt + \frac{i}{2\pi i} \int_0^{-\infty} \frac{g(it)e^{-iait}}{it-z} dt. \quad (21)$$

Using

$$\frac{1}{z-it} = \int_0^1 s^{z-it-1} ds \quad \text{and} \quad \int_{-\alpha}^{\alpha} e^{i\theta(t-z)} i d\theta = \frac{e^{i\alpha(it-z)} - e^{-i\alpha(it-z)}}{t-z}$$

(20) multiplied by $e^{-\alpha zi}$ plus (21) multiplied by $e^{\alpha zi}$, we obtain, for $z = x + iy$, $x > 0, y > 0$,

$$g(z) = \frac{1}{2\pi} \int_0^{+\infty} g(t) \int_{-\alpha}^{\alpha} e^{i\theta(t-z)} d\theta dt - \frac{1}{2\pi} \int_{-\infty}^0 g(it) \int_0^1 (se^{i\alpha})^{(z-it)} \frac{ds}{s} dt - \frac{1}{2\pi} \int_0^{+\infty} g(it) \int_0^1 (se^{-i\alpha})^{(z-it)} \frac{ds}{s} dt. \quad (22)$$

Similarly, (22) also holds for $x > 0, y < 0$. The interchange in order of integration in (22) are legitimate: in the integrands in (22) are replaced by their absolute values, some finite integral results. Hence (22) can be rewritten in the form

$$g(z) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} e^{i\theta z} h_0(e^{i\theta}) d\theta + \frac{1}{2\pi i} \int_0^1 ((se^{-i\alpha})^z h_1(se^{-i\alpha}) - (se^{i\alpha})^z h_{-1}(se^{i\alpha})) \frac{ds}{s},$$

where

$$h_l(\zeta) = \int_{L_l} g(z) \zeta^{-z} dz \quad (23)$$

and $L_l = \{t \exp\{\frac{\pi}{2}li\} : t \geq 0\}$ ($l \in \{-1, 0, 1\}$) are half-lines. By (21), $h_0(\zeta)$ is analytic in the region $D_0 = \{\zeta : |\zeta| > 1, |\arg \zeta| < \pi\}$ and continuous in the set $\overline{D_0} = \{\zeta : |\zeta| \geq 1, |\arg \zeta| < \pi\}$, each function $h_l(\zeta)$ ($l = \pm 1$) is analytic in the sector $D_l = \{\zeta : \alpha < -l \arg \zeta < \pi\}$ and continuous in the closure $\overline{D_l} = \{\zeta : \alpha \leq -l \arg \zeta \leq \pi\}$ of D_l . By Cauchy's formula, $h_0(\zeta)$ can be continued analytically to a bounded analytic function in the region $D_{-1} \cup D_0 \cup D_1 = \{\zeta = \rho e^{i\phi} : \zeta \notin I_\alpha, |\phi| < \pi\}$, i.e., $h_0(\rho e^{i\phi}) = h_l(\rho e^{i\phi})$ for $\rho > 1, \alpha < -l\phi < \pi, l = \pm 1$. By (21), $h_0(\zeta)$ is bounded in the circular arc $\{\zeta : |\zeta| = 1, |\arg \zeta| < \alpha\}$. Integrations by parts twice in (23),

$$h_l(se^{-il\alpha}) = (\log s - il\alpha)^{-2} \int_{L_l} g''(z) (se^{-il\alpha})^{-z} dz, \quad l = \pm 1.$$

By (19),

$$\int_0^1 (|h_{-1}(se^{i\alpha})| + |h_1(se^{-i\alpha})|) \frac{ds}{s} < \infty.$$

Therefore, the linear functional

$$T(\varphi) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \varphi(e^{i\theta}) h_0(e^{i\theta}) d\theta + \frac{1}{2\pi i} \int_0^1 (\varphi(se^{-i\alpha}) h_1(se^{-i\alpha}) - \varphi(se^{i\alpha}) h_{-1}(se^{i\alpha})) \frac{ds}{s}$$

is a bounded linear functional on H_α and satisfies $T(\zeta^\lambda) = g(\lambda)$ for $\lambda \in \mathbb{C}_+$. By the Riesz representation theorem, $z^\lambda \notin \overline{\text{span}} E(\Lambda)$ for $\lambda \notin \Lambda$ and $\text{Re } \lambda > 0$. Similarly, replacing $g(z)$ by $(z - \lambda)^{-1}g(z)$ for $\lambda \in \Lambda$, we can also prove that no element of $E(\Lambda)$ belongs to the closure of the vector subspace generated by the other elements of $E(\Lambda)$. Therefore, $E(\Lambda)$ is minimal, and $\overline{\text{span}} M(\Lambda) \neq H_\alpha$.

Next, define, for $0 < b = \frac{\alpha}{\pi} < \infty$, the arithmetic progression [13] Λ_b by

$$\Lambda_b = \left\{ \frac{n}{b} : n = 1, 2, \dots \right\}$$

and observe that the counting function $\Lambda_b(t) = \sum_{n \leq b} 1$ of Λ_b satisfies $\Lambda_b(t) = [bt] = bt + O(1)$, and the characteristic logarithm $\lambda_b(t)$ of the sequence Λ_b satisfies

$$\lambda_b(t) = b \log t + b \log b + b\gamma + O(t^{-1}),$$

as $t \rightarrow \infty$, where γ is a Euler constant. So by Lemma 2, there exist a constant A_1 and a subsequence $\Lambda^* = \{\lambda_n^* : n = 1, 2, \dots\}$ of Λ_b such that (9) holds. If Λ and Λ^* have common elements or $\delta(\Lambda \cup \Lambda^*) = 0$, we adjust Λ^* as follows: let $4h_1 = \min\{\delta(\Lambda), b\}$ and $n_k \in \mathbb{N}$ such that $\lambda_{n_k} \leq \lambda_k^* \leq \lambda_{1+n_k}$ and let

$$\lambda_k^{**} = \begin{cases} \lambda_k^*, & \text{if } \lambda_{n_k} + h_1 \leq \lambda_k^* < \lambda_{1+n_k} - h_1; \\ \lambda_k^* + h_1, & \text{if } \lambda_{n_k} \leq \lambda_k^* < \lambda_{n_k} + h_1; \\ \lambda_k^* - h_1, & \text{if } \lambda_{1+n_k} - h_1 < \lambda_k^* < \lambda_{1+n_k}, \end{cases}$$

and let

$$A_3 = \sum_{k=1}^{+\infty} \left(\frac{1}{\lambda_k^*} - \frac{1}{\lambda_k^{**}} \right),$$

then the set $\Lambda^{**} = \{\lambda_n^{**} : n = 1, 2, \dots\}$ and the set Λ are disjoint and $\delta(\Lambda \cup \Lambda^{**}) \geq h_1 > 0$. For $x \geq 2h_1 + 1$, we have the following inequalities:

$$|\lambda^*(x) - \lambda^{**}(x) - A_3| \leq \frac{1}{x} + \sum_{\lambda_k^* \geq x} \frac{h_1}{\lambda_k \lambda_k^*} \leq \frac{1}{x} + \frac{1}{x - h_1} \leq \frac{3}{x}$$

and

$$|\lambda(x) + \lambda^{**}(x) - \frac{\alpha}{\pi} \log x - A_1 - A_3| \leq \frac{13}{x} + \varepsilon(x).$$

Suppose that f is in $\overline{\text{span}}[E(\Lambda)]$, since $\overline{\text{span}}[E(\Lambda)] \subset \overline{\text{span}}[E(\Lambda \cup \Lambda^{**})]$, then from the uniqueness of $E(\Lambda \cup \Lambda^{**})$ -expansion $\sum b_n z^{\mu_n}$ of f , where $\Lambda \cup \Lambda^{**} = \{\mu_n : n = 1, 2, \dots\}$, we see that those coefficients b_n associated with members $\mu_n \in \Lambda^{**}$ distinct from all λ_n are equal to zero. Thus the $E(\Lambda \cup \Lambda^{**})$ -expansion reduces to $E(\Lambda)$ -expansion whenever $f \in \overline{\text{span}} E(\Lambda)$. Therefore, we can assume, without loss of generality, that there exists a constant A_2 such that (11) holds with $b = \frac{\alpha}{\pi}$. Therefore, the function $g_0(z)$ defined by (12) satisfies (13), (14) and (15). Let

$$\psi_k(z) = \frac{z^2 g_0(z)}{(1+z)^4 (z - \lambda_k)}, \quad \psi_k(\lambda_k) = \frac{\lambda_k^2 g_0'(\lambda_k)}{(1 + \lambda_k)^4},$$

and

$$h_{k,l}(\zeta) = \int_{L_l} \psi_k(z) \zeta^{-z} dz,$$

where $L_l = \{t \exp\{\frac{\pi}{2}li\} : t \geq 0\}$ ($l \in \{-1, 0, 1\}$) are half-lines. As has been shown in (19), then there exists a positive constant A_4 such that

$$|\psi_k(iy)| + |\psi_k'(iy)| + |\psi_k''(iy)| \leq \frac{A_4 e^{\alpha|y|}}{1 + |y|^2} \quad (24)$$

and

$$\limsup_{x \rightarrow \infty} x^{-1} \log |\psi_k(x)| = 0 \quad (25)$$

hold for each k . By (15),

$$\limsup_{k \rightarrow \infty} \lambda_k^{-1} \log |\psi_k(\lambda_k)| = 0. \quad (26)$$

By (24) and (25), $h_{k,0}(\zeta)$ is analytic in the region $D_0 = \{\zeta : |\zeta| > 1, |\arg \zeta| < \pi\}$, $h_{k,l}(\zeta)$ ($l = \pm 1$) is analytic in the sector $D_l = \{\zeta : \alpha < -l \arg \zeta < \pi\}$ and continuous in the closure $\overline{D_l} = \{\zeta : \alpha \leq -l \arg \zeta \leq \pi\}$ of D_l . By Cauchy's formula, $h_{k,0}(\zeta)$ can be continued analytically to an analytic function in the region $D_{-1} \cup D_0 \cup D_1 = \{\zeta = \rho e^{i\phi} : \zeta \notin I_\alpha, |\phi| < \pi\}$, i.e., $h_{k,0}(\rho e^{i\phi}) = h_{k,l}(\rho e^{i\phi})$ for $\rho > 1, \alpha < -l\phi < \pi, l = \pm 1$. By (25), $h_0(e^{-\delta}\zeta)$ is bounded in the circular arc $\{\zeta : |\zeta| = 1, |\arg \zeta| < \alpha\}$ for each $\delta > 0$. the linear functionals

$$\begin{aligned} T_{k,\delta}(\varphi) &= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \varphi(e^{i\theta}) h_{k,0}(e^{-\delta} e^{i\theta}) d\theta \\ &+ \frac{1}{2\pi i} \int_0^1 (\varphi(se^{-i\alpha}) h_{k,1}(e^{-\delta} se^{-i\alpha}) - \varphi(se^{i\alpha}) h_{k,-1}(e^{-\delta} se^{i\alpha})) \frac{ds}{s} \quad (\varphi \in H_\alpha) \end{aligned}$$

are bounded linear functionals in H_α and satisfy $T_{k,\delta}(\zeta^\lambda) = \psi_k(\lambda) e^{-\delta\lambda}$ for $\lambda \in \mathbb{C}_+$ and $A(\delta) = \sup\{|T_{k,\delta}| : k = 0, 1, 2, \dots\} < \infty$. Therefore, $\{e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1} T_{k,\delta} : k = 1, 2, \dots\}$ is a biorthogonal system of $E(\Lambda)$. If f belongs to $\overline{\text{span}}[E(\Lambda)]$, then there exists a sequence of Λ -polynomials

$$P_l(z) = \sum_{n=1}^l a_{n,l} z^{\lambda_n} \in \text{span} E(\Lambda)$$

such that

$$\|f - P_l\| \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

Let

$$\sum_{k=1}^{\infty} a_k z^{\lambda_k} \quad (27)$$

be the $E(\Lambda)$ -expansion of f . The biorthogonality of the system

$$\{e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1} T_{k,\delta} : k = 1, 2, \dots\}$$

implies that $a_k = e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1} T_{k,\delta}(f)$ and $a_{k,l} = e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1} T_{k,\delta}(P_l)$. Therefore

$$|a_k - a_{k,l}| \leq \|f - P_l\| A(\delta) e^{\delta\lambda_k} |\psi_k(\lambda_k)|^{-1} \quad (k = 1, 2, \dots)$$

(thus that the sequences $\{a_{k,l}\}$ are independent of δ implies that the sequence $\{a_k\}$ is also independent of δ) and

$$|a_k| \leq A(\delta) \|f\| e^{\delta\lambda_k} |\psi_k(\lambda_k)|^{-1}, \quad k = 1, 2, \dots$$

By (26), the series in (27) converges to an analytic function $F(z)$ uniformly on compacts of $\{z : |z| < 1, |\arg z| < \pi\}$. we obtain that, for $z \in \text{int}[I_\alpha]$, there is $\delta > 0$ such that $|z| < e^{-\delta}$, so

$$\begin{aligned} |f(z) - F(z)| &\leq |f(z) - P_l(z)| + |P_l(z) - F(z)| \\ &\leq \|f - P_l\| + \sum_{n=1}^l |a_{nl} - a_n| |z|^{\lambda_n} + \sum_{n=l+1}^{\infty} |a_n| |z|^{\lambda_n}. \end{aligned}$$

Letting $l \rightarrow \infty$, we obtain that $f(z) = F(z)$ for $z \in \text{int} I_\alpha$. This completes the proof of Theorem.

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